NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY
251 Mercer St. New York, N.Y. 10012

IMM 350
June 1966

Courant Institute of
Mathematical Sciences

On the Asymptotic Normality of One Sample Chernoff-Savage Test Statistics

Madan Lal Puri and Pranab Kumar Sen

Prepared under Contract Nonr-285(38) with
the Office of Naval Research NR 042-206,
and Research Grant GM-12868
with the National Institutes of Health,
Public Health Service

Distribution of this document is unlimited.

New York University

IMM-350
c.1

NR 042-206

IMM 350
June 1966

New York University
Courant Institute of Mathematical Sciences

ON THE ASYMPTOTIC NORMALITY OF ONE SAMPLE CHERNOFF-SAVAGE TEST STATISTICS

Madan Lal Puri[1] and Pranab Kumar Sen[2]

[1] This report represents results obtained in part while this author was at the University of North Carolina, Chapel Hill.

[2] Presently at the University of North Carolina, Chapel Hill. On leave of absence from the University of Calcutta.

This work was partially supported by the Office of Naval Research, Contract Nonr-285(38), and the National Institutes of Health, Public Health Service, Grant GM-12868.
Reproduction in whole or in part is permitted for any purpose of the United States Government.

Distribution of this document is unlimited.

C.1

Asymptotic normality of a class of one sample Chernoff-Savage [1] type of rank order statistics has been established by Govindarajulu [2]. His proof appears to be quite lengthy and cumbrous. The object of this note is to provide a greatly shortened and simplified proof of the same theorem.

1. INTRODUCTION

Let $X_1, \ldots, X_n$ be n independent and identically distributed random variables (i.i.d.r.v.) having an absolutely continuous cumulative distribution function (cdf) F(x), defined on the real line $\{-\infty < x < \infty\}$. Let $Z_{n,\alpha}$ be equal to 1 if the α-th smallest observation among the values of $|X_i|$, $i = 1, \ldots, n$, is from a positive X and let $Z_{n,\alpha}$ be equal to zero, otherwise, (for $\alpha=1,\ldots,n$). By virtue of the assumed continuity of F(x), the possibilities of ties among the observations $|X_i|$, $i=1,\ldots,n$ and of some of them being exactly equal to zero may be ignored, in probability. We shall be concerned with the Chernoff-Savage [1] type of one sample rank order statistics which may be expressed as

$$T_n = \frac{1}{n} \sum_{\alpha=1}^{n} Z_{n,\alpha} E_{n,\alpha} , \tag{1.1}$$

where $E_{n,\alpha} = J_n(\alpha/(n+1))$, $\alpha=1,\ldots,n$ are functions of the ranks $\alpha(=1,\ldots,n)$ and are explicitly known. We are interested in the asymptotic normality of the standardized form of T_n, for any arbitrary F(x). The only theorem of this note will be presented in the next section, while here we consider the preliminary notions and regularity conditions.

The function $J_n(u)$ is defined only at $u = \frac{\alpha}{n+1}$, $\alpha=1,\dots,n$, but we may extend its domain of definition to $(0,1)$ by letting it have constant values over $[\frac{\alpha}{n+1}, \frac{\alpha+1}{n+1})$, $\alpha=1,\dots,n$. Further, we define

$$F_n(x) = \frac{1}{n}\,(\text{Number of } X_\alpha \leq x); \tag{1.2}$$

$$H_n(x) = \frac{1}{n}\,(\text{Number of } |X_\alpha| \leq x) = F_n(x) - F_n(-x); \tag{1.3}$$

$$H(x) = F(x) - F(-x). \tag{1.4}$$

Concerning $J_n(u)$ we then assume (following [1,3]) that

(I) $\lim_{n\to\infty} J_n(u) = J(u)$ exists for $0 < u < 1$ and is not a constant;

(II) $\int_0^\infty \{J_n[\frac{n}{n+1} H_n(x)] - J[\frac{n}{n+1} H_n(x)]\}\, dF_n(x) = o_p(n^{-\frac{1}{2}})$; and

*(III) $J(u)$ is absolutely continuous for $0 < u < 1$ and

$$|J^{(i)}(u)| = \left|\frac{d^i}{du^i} J(u)\right| \leq K\{u(1-u)\}^{-\frac{1}{2}-i+\delta}, \quad \text{for } i=0,1,2,$$

where K is some constant and $\delta > 0$.

2. THE MAIN THEOREM

We rewrite T_n as

$$T_n = \int_0^\infty J_n[\frac{n}{n+1} H_n(x)]\, dF_n(x), \tag{2.1}$$

---

* In the proof of the main theorem we shall not use the second derivative of $J(u)$. However, this condition appears to be essential for the proper use of some well-known scores (e.g., normal scores etc.), and hence, we prefer to include this condition explicitly, with a note that for some simple scores this is not essential.

and define

$$(2.2)\qquad \mu_n = \int_0^\infty J[H(x)]\, dF(x);$$

$$(2.3)\qquad \sigma_n^2 = 2\Big[\iint_{0<x<y<\infty} F(x)[1-F(y)]\, J'[H(x)]\, J'(H(y)]\, dF(-x)\, dF(-y)$$

$$+ \iint_{0<x<y<\infty} F(-y)[1-F(-x)]\, J'[H(x)]\, J'[H(y)]\, dF(x)\, dF(y)$$

$$- \int_0^\infty \int_0^\infty F(-x)[1-F(y)]\, J'[H(x)]\, J'[H(y)]\, dF(x)\, dF(-y)\Big].$$

THEOREM 2.1. *For any absolutely continuous cdf F(x), under the conditions (I), (II) and (III),*

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}[T_n - \mu_n]/\sigma_n \leq x\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt,$$

provided $\sigma_n \neq 0$.

PROOF. Writing $F_n(x) = F(x) + [F_n(x) - F(x)]$, making use of condition (II) and finally by expansion of $J(\frac{n}{n+1} H_n(x)]$ around $J[H(x)]$, we get after little simplification that

$$(2.4)\qquad T_n = \mu_n + B_{1n} + B_{2n} + \sum_{i=1}^{4} C_{in},$$

where μ_n is defined by (2.2) and

$$(2.5)\qquad B_{1n} = \int_0^\infty J[H(x)]\, d[F_n(x) - F(x)],$$

$$(2.6)\qquad B_{2n} = \int_0^\infty [H_n(x) - H(x)]\, J'[H(x)]\, dF(x),$$

$$(2.7)\qquad C_{1n} = \frac{-1}{n+1} \int_0^\infty H_n(x)\, J'[H(x)]\, dF_n(x),$$

(2.8) $$C_{2n} = \int_0^\infty [H_n(x) - H(x)]\, J'[H(x)]\, d[F_n(x) - F(x)],$$

(2.9) $$C_{3n} = \int_0^\infty \{J[\tfrac{n}{n+1} H_n(x)] - J[H(x)] - [\tfrac{n}{n+1} H_n(x) - H(x)]\, J'[H(x)]\}\, dF_n(x),$$

(2.10) $$C_{4n} = \int_0^\infty \{J_n[\tfrac{n}{n+1} H_n(x)] - J[\tfrac{n}{n+1} H_n(x)]\} dF_n(x).$$

The term μ_n is finite by condition (III), while C_{in} $(i=1,2,3,4)$ are all $o_p(n^{-\frac{1}{2}})$, the proof being exactly same as in [3]. Thus, we require only to show that $n^{\frac{1}{2}}[B_{1n} + B_{2n}]$ has asymptotically a normal distribution with zero mean and variance σ_n^2, defined by (2.3). Now, integrating B_{1n} by parts and adding B_{2n} to it, we obtain

(2.11) $$\begin{aligned} B_{1n} + B_{2n} &= \int_0^\infty [F_n(x) - F(x)] J'[H(x)]\, dF(-x) \\ &\quad - \int_0^\infty [F_n(-x) - F(-x)]\, J'[H(x)]\, dF(x). \\ &= \frac{1}{n} \sum_{\alpha=1}^{n} B(X_\alpha), \text{(Say,)}, \end{aligned}$$

where

(2.12) $$B(X) = \int_0^\infty [F_1(x)-F(x)] J'[H(x)] dF(-x) - \int_0^\infty [F_1(-x)-F(-x)] J'[H(x)] dF(x),$$

$F_1(x)$ being the empirical cdf in a sample of size 1. By an application of Fubini's theorem (cf [4], p. 135), it is easily seen that

(2.13) $$E_F[B(X)] = 0 \text{ and } E_F[B(X)]^2 = \sigma_n^2, \quad \text{defined by (2.3)}$$

Also, by condition (III), $E_F|B(X)|^{2+\delta} < \infty$, for some $\delta > 0$, and hence $\sigma_n^2 < \infty$. Since σ_n^2 is assumed to be positive, and by (2.11), $B_{1n} + B_{2n}$ is the average of n i.i.d.r.v.'s having zero mean and variance $\sigma_n^2 > 0$, the asymptotic normality of $n^{\frac{1}{2}}[B_{1n} + B_{2n}]$ readily follows from the classical central limit theorem, under Lindeberg's condition (cf. [4], p. 280).

Hence, the theorem.

COROLLARY 2.1. *If (i) the assumptions of theorem 2.1 hold,* (ii) $F(x)$ *is replaced by a sequence of cdf's* $\{F_{(n)}(x)\}$, *where* $F_{(n)}(x)$ *is symmetric about* $\theta/n^{\frac{1}{2}}$, θ *being real and finite,* (iii) *the function* $J_n(\alpha/(n+1))$ *is the expected value of the α-th order statistic in a sample of size* n *from a distribution* $\psi(x)$, *where*

$$(2.14) \qquad \psi(x) = \psi^*(x) - \psi^*(-x), \quad x \geq 0,$$

$\psi^*(x)$ *being symmetric about* $x=0$, then

$$(2.15) \qquad \lim_{n\to\infty} \sigma_n^2 = \tfrac{1}{4} \int_0^1 J^2(u)\, du.$$

The proof is an immediate consequence of theorem 2.1, and therefore is omitted.

It is clearly seen how much the proof of the main theorem is simplified as compared to the one in [2].

REFERENCES

[1] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *The Annals of Mathematical Statistics*. *29*, 972-994.

[2] Govindarajulu, A. (1960). Central limit theorems and asymptotic efficiency for one sample nonparametric procedures. *Technical report no. 11, Department of Statistics, University of Minnesota*.

[3] Govindarajulu, Z., LeCam, L. and Raghavachari, M. (1965). Generalizations of Chernoff-Savage theorems on asymptotic normality of nonparametric test statistics. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*. *1* (in Press).

[4] Loeve, M. (1963). *Probability theory*. D. Van Nostrand Co., New York.

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Courant Institute of Mathematical Sciences New York University	2a. REPORT SECURITY CLASSIFICATION not classified
	2b GROUP none

3. REPORT TITLE

On the Asymptotic Normality of One Sample
Chernoff-Savage Test Statistics

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report June 1966

5. AUTHOR(S) *(Last name, first name, initial)*

Puri, Madan Lal and Sen, Pranab Kumar

6. REPORT DATE June 1966	7a. TOTAL NO. OF PAGES 6	7b. NO. OF REFS 4
8a. CONTRACT OR GRANT NO. Nonr-285(38) b. PROJECT NO. NR 042-206 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) IMM 350 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* none	

10. AVAILABILITY/LIMITATION NOTICES

Qualified requesters may obtain copies of this report from DDC.

11. SUPPLEMENTARY NOTES none	12. SPONSORING MILITARY ACTIVITY U.S. Navy, Office of Naval Research 207 West 24th St., New York, N.Y.

13. ABSTRACT

Asymptotic normality of a class of one sample Chernoff-Savage type of rank order statistics has been established by Govindarajulu [(1960), Central limit theorems and asymptotic efficiency for one sample nonparametric procedures, Technical report no. 11, Department of Statistics, University of Minnesota]. His proof appears to be quite lengthy and cumbersome. The object of this note is to provide a greatly shortened and simplified proof of the same theorem.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

BASIC DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL REPORTS

Head, Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D.C. 20360	3 copies
Commanding Officer Office of Naval Research Branch Office Navy #100 Fleet Post Office New York, New York	2 copies
Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20 copies
Defense Logistics Studies Information Exchange Army Logistics Management Center Fort Lee, Virginia Attn: William B. Whichard	1 copy
Technical Information Officer Naval Research Laboratory Washington, D.C. 20390	6 copies
Commanding Officer Office of Naval Research Branch Office 207 West 24th Street New York, New York 10011 Attn: J. Laderman	1 copy
Commanding Officer Office of Naval Research Branch Office 1030 East Green Street Pasadena 1, California Attn: Dr. A.R. Laufer	1 copy
Bureau of Supplies and Accounts Code OW Department of the Navy Washington 25, D.C.	1 copy
Institute for Defense Analyses Communications Research Division von Neumann Hall Princeton, New Jersey	1 copy
University of Chicago Statistical Research Center Chicago, Illinois Attn: Prof. Paul Meier	1 copy

Stanford University
Applied Mathematics & Statistics Lab.
Stanford, California
Attn: Prof. Gerald J. Lieberman

Florida State University
Department of Statistics
Tallahassee, Florida
Attn: Prof. I.R. Savage

Florida State University
Department of Statistics
Tallahassee, Florida
Attn: Dr. Ralph A. Bradley

Princeton University
Department of Mathematics
Princeton, New Jersey

Columbia University
Department of Mathematical Statistics
New York 27, New York
Attn: Prof. T.W. Anderson

University of California
Department of Statistics
Berkeley 4, California
Attn: Prof. J. Neyman

University of Washington
Department of Mathematics
Seattle 5, Washington
Attn: Prof. Z.W. Birnbaum

Cornell University
Department of Mathematics
Ithaca, New York
Attn: Prof. J. Wolfowitz

Harvard University
Department of Statistics
Cambridge, Massachusetts
Attn: Prof. W.G. Cochran

The Research Triangle Institute
Statistics Research Division
505 West Chapel Hill Street
Durham, North Carolina
Attn: Dr. Malcolm R. Leadbetter

Columbia University
Dept. of Industrial Engineering
New York 27, New York
Attn: Prof. Cyrus Derman

Columbia University
Department of Mathematics
New York 27, New York
Attn: Prof. H. Robbins

New York University
Courant Inst. of Mathematical Sciences
New York , New York 10012
Attn: Prof. W.M. Hirsch

Cornell University
Department of Plant Breeding
Biometrics Unit
Ithaca, New York
Attn: Walter T. Federer

University of North Carolina
Statistics Department
Chapel Hill, North Carolina
Attn: Prof. Walter L. Smith

Stanford University
Department of Statistics
Stanford, California
Attn: Prof. Herman Rubin

Brown University
Division of Applied Mathematics
Providence 12, Rhode Island
Attn: Prof. M. Rosenblatt

New York University
Department of Industrial Engineering
and Operations Research
Bronx 63, New York
Attn: Prof. J. H. Kao

University of Wisconsin
Department of Statistics
Madison, Wisconsin
Attn: Prof. G.E.P. Box

Rocketdyne - A Division of Nort
American Aviation, Inc.
6633 Canoga Avenue
Canoga Park, California

Attn: Dr. Robert Zimmerman

Prof. Anders Hald
Institute of Mathematical Statistics
University of Copenhagen
Copenhagen, Denmark

Yale University
Department of Mathematics
New Haven, Connecticut
Attn: Prof. L.J. Savage

Massachusetts Institute of Technology
Department of Electrical Engineering
Cambridge, Massachusetts
Attn: Dr. R.A. Howard

Stanford University
Department of Statistics
Stanford, California
Attn: Prof. E. Parzen

Arcon Corporation
803 Massachusetts Avenue
Lexington 73, Massachusetts
Attn: Dr. Arthur Albert

University of California
Institute of Engineering Research
Berkeley 4, California
Attn: Prof. R.E. Barlow

Michigan State University
Department of Statistics
East Lansing, Michigan
Attn: Prof. J. Gani

Dr. Jack R. Borsting
Chairman
Dept. of Operations Analysis
U. S. Naval Postgraduate Sch.
Monterey, California 93940

Prof. Harold E. Dodge
Rutgers - The State University
Statistics Center
New Brunswick, New Jersey

Yale University
Department of Statistics
New Haven, Connecticut
Attn: Prof. F.J. Anscombe

Purdue University
Division of Mathematical Sciences
Lafayette, Indiana
Attn: Prof. S.S. Gupta

Cornell University
Department of Industrial Engineering
Ithaca, New York
Attn: Prof. Robert Bechhofer

Stanford University
Department of Statistics
Stanford, California
Attn: Prof. C. Stein

Applied Mathematics and Statistics Lab.
Department of Statistics
Stanford University
Stanford, California
Attn: Prof. H. Solomon

Decision Studies Group
460 California Avenue
Palo Alto, California
Attn: Warren R. Ketler

The Johns Hopkins University
Department of Mathematical Statistics
34th and Charles Streets
Baltimore 18, Maryland
Att: Prof. Geoffrey S. Watson

OCT 1 3 1966

DATE DUE

JAN 2 6 19

GAYLORD PRINTED IN U.S.A.

NYU
IMM-
350
Puri c.1

On the asymptotic normality of one sample Chernoff-Savage test statistics.

NYU
IMM-
350 c.1
Puri

AUTHOR On the asymptotic normality of one sample

TITLE Chernoff-Savage test statistics.

DATE DUE	BORROWER'S NAME
JAN 22 1964	Parris

N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York 12, N. Y.